Second Workshop on Hyperbolic Problems: Theory, Numerics and Applications

Universidad de La Frontera, sede Angol, January 8 and 9, 2026

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Organizadores:

- **Raimund Bürger**, Universidad de Concepción,
- **Gino Montecinos**, Universidad de La Frontera,
- **Luis-Miguel Villada**, Universidad del Bío-Bío.

Patrocinadores:

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Second Workshop on Hyperbolic Problems: Theory, Numerics and Applications

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11.30–12.00 **Cristóbal E. Castro** (Universidad de Tarapacá, Arica, Chile):

ADER schemes for geophysical problems: well balanced, local time step and subcell resolution

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Variants for the WENO-DK reconstruction of even orders for hyperbolic PDEs

12.30 **Closure**

A novel Lagrangian-Eulerian approach for nonlinear hyperbolic balance laws: theory, computing and applications

Eduardo Abreu *
Universidade Estadual de Campinas (UNICAMP) - IMECC
Department of Applied Mathematics, Brazil

Abstract

In this work, a new numerical Lagrangian-Eulerian approach for nonlinear hyperbolic balance laws is presented with applications, as recently introduced in [1]. Here, the novel Lagrangian-Eulerian approach refers to a class of numerical methods for solving hyperbolic/balance hyperbolic laws, which combine both, a Lagrangian step (following space-time points via the novel concept of *no-flow curves* [2]), and an Eulerian step (a remap at fixed spatial grid) perspectives to accurately simulate phenomena like shock waves, particularly in problems where the mesh needs to adapt to large deformations or complex flows; see, e.g., [3]. This new approach offers a more robust and accurate way to handle the conservation properties of hyperbolic equations and the complexities of their source terms. We will discuss the new ideas of such novel class of Lagrangian–Eulerian schemes [1] for the 1D model problem (but we will also present new 2D results) for $u(x,t)\colon \mathbb{R}\times\mathbb{R}_+ \to \Omega \subset \mathbb{R}$,

$$\frac{\partial}{\partial t}A(u(x,t))+\frac{\partial}{\partial x}F(u(x,t)) = G(u(x,t)), \quad x\in\mathbb{R}, t\in\mathbb{R}^+, t>0,\ u(x,0)=u_0(x), \quad x\in\mathbb{R}, \tag{1}$$

where $u_0 \in L^\infty(\mathbb{R})\cap BV(\mathbb{R})$ and A, F are assumed to be twice continuously differentiable functions; A is also invertible, and the source term G is uniformly bounded, but nonlinear and possibly *stiff* [5] The dynamics of nonlinear hyperbolic balance laws (1), typically under effects of stiff source term, is of uppermost importance in the several disciplines of fluid mechanics [3] and reaction transport problems in porous media [5, 2]. We developed both fully discrete and semi-discrete formulations, and extended the concept of *no-flow curves* [2, 4] to this general class of nonlinear balance laws. We established a rigorous numerical entropic-convergence study using weak asymptotic analysis, which involved investigating the existence, uniqueness, and regularity of entropy-weak solutions computed with our scheme [1]. To evaluate the shock capturing capabilities of the new Lagrangian-Eulerian numerical scheme, we carried out numerical experiments that demonstrate its ability to accurately resolve the key features of balance laws [4, 1] and hyperbolic problems [2, 4]. The proposed method is Riemann-solver-free and subject to a novel *weak CFL* stability condition [4, 5].

This presentation is based on joint research with E. Pandini (UDESC - Universidade do Estado de Santa Catarina) and W. Lambert (UNIFAL-MG - Universidade Federal de Alfenas, Brazil).

*This author was supported by CNPq (307641/2023-6) and FAPESP (2025/07662-6). E-mail: eabreu@unicamp.br

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A second-order invariant-region-preserving scheme for a transport-flow model of polydisperse sedimentation

Raimund Bürger *
CI²MA and Departamento de Ingeniería Matemática
Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción
Casilla 160-C, Concepción, Chile

Abstract

A polydisperse suspension is a mixture of a number N of species of small solid particles, which may differ in size or density, dispersed in viscous fluid. The sedimentation of such a mixture gives rise to the segregation of species and flow of the mixture due to density fluctuations. In two space dimensions, and for equal-density particles, this process can be described by a hyperbolic system of N nonlinear conservation laws for the particle volume fractions coupled with a version of the Stokes system for the volume-averaged flow field of the mixture [1, 2, 4, 5]. A second-order numerical scheme for this transport-flow model is formulated by combining a finite-difference approximation of the Stokes system with a finite volume (FV) scheme for the transport equations, both defined on a Cartesian grid on a rectangular domain. The FV scheme is based on a central weighted essentially non-oscillatory (CWENO) reconstruction [3] applied to the first-order local Lax-Friedrichs (LLF) numerical flux. By the application of scaling limiters to the CWENO reconstruction polynomials (following [6, 7, 8]) and utilizing that the Stokes solver generates a discretely divergence-free (DDF) velocity field, one can prove that the FV scheme has the invariant region preserving (IRP) property, i.e., the volume fractions are nonnegative and sum up at most to a set maximum value. Numerical examples illustrate the model and the scheme.

This presentation is based on joint research with Juan Barajas-Calonge and Luis Miguel Villada (Universidad del Bío-Bío, Chile) and Pep Muleyt(Universitat de València, Burjassot, Spain).

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Inverse problem for the Shallow Water equations in the Bío Bío river geomorphological context

Fernando Campos*
Departamento de Matemática
and CI^{2}MA -Universidad de Concepción

Abstract

In this talk, we present a method for parameters identification for the Shallow Water system. In particular, we are interested in identifying the bathymetry and the soil's friction coefficient from satellite observations made by the SWOT mission. The method uses a gradient method to minimize a cost function that includes a numerical scheme for the one-dimensional Shallow Water equations. Usually, for the one-dimensional problem is used a finite difference scheme [1] or simplified equations instead of the full Shallow Water system [2]. In our case, we consider a second-order finite volume scheme with slope limiters. We use a Lagrangian method to obtain an adjoint scheme associated to this numerical scheme. To avoid errors, we use the automatic differentiation tool TAPENADE [3] to calculate the adjoint scheme and the gradient for the cost function. We test this method with the results from [4] and we compare the calculated gradient with a simple secant method, obtaining relative errors no greater than 10^{-7}. We use this method to identify a simple bathymetry from the results in [4]. As we think that sediment transport is a relevant factor in the Bío Bío river, we also developed this method for the Saint-Venant-Exner model. We use the three-wave approximate Riemann solver shown in [5] and test the method with the numerical results presented therein to identify the initial bathymetry.

This presentation is based on joint research with Mauricio Sepúlveda, Rodrigo Abarca-del-Río and Cristóbal Caro (UdeC-Chile).

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*This author was supported by ANID (Chile). E-mail: `fcamposg@udec.cl`

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ADER schemes for geophysical problems: Well balanced, local time step and subcell resolution.

Cristóbal E. Castro *
Departamento de Ingeniería Mecánica, Universidad de Tarapacá

Abstract

In this talk, we revisit previous work in which key extensions to the ADER framework were introduced in the context of Finite Volume and Discontinuous Galerkin methods for geophysical applications such as tsunami and seismic wave propagation. For tsunami simulations based on the shallow water equations, the continuous equilibrium states inherent to the mathematical model are not naturally preserved by standard finite volume schemes on unstructured meshes. When a numerical method preserves this equilibrium at the discrete level, it is referred to as well balanced [1].

In this work, we propose a well-balanced time expansion of the reconstructed free surface, enabling accurate simulation of tsunami wave propagation in realistic scenarios. This equilibrium condition was previously analyzed in [2], where numerical errors associated with the loss of balance were shown to decrease significantly with increasing order of accuracy; however, this alone is insufficient for realistic tsunami events.

Another important aspect in geophysical simulations concerns the presence of multi-scale temporal and spatial features. In such settings, slow-moving waves can suffer from excessive numerical diffusion when global time-stepping strategies are employed. In [3], we introduced a clustering approach in which a finite set of time scales is used to group finite volumes, thereby reducing numerical diffusion.

Finally, in the context of ADER-DG applied to the linear elastic equations for seismic wave propagation, highly heterogeneous materials may require very fine mesh resolutions to accurately capture spatial variability. In [4], we demonstrated that decoupling the high-order representation of the unknown fields from that of the material coefficients can lead to faster numerical algorithms while preserving accuracy

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*This author was supported by ANID (Chile) through Centro de Modelamiento Matemático (CMM; BASAL project FB210005). E-mail: `ccastro@academicos.uta.cl`

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Second Workshop on Hyperbolic Problems: Theory, Numerics and Applications

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Hyperbolic Aspects in the Numerical Modeling of PM2.5 Transport Using a Reaction–Diffusion–Advection Equation

Joaquín González Monsalves*
Departamento de Ingeniería Matemática, Universidad de La Frontera, Chile
In collaboration with
Gino Montecinos†
Departamento de Ingeniería Matemática, Universidad de La Frontera, Chile

Abstract

Air pollution caused by fine particulate matter (PM2.5) represents one of the main environmental and public health concerns in southern Chile, particularly in Temuco, where residential wood combustion leads to severe winter episodes.

The physical process behind PM2.5 formation and dispersion couples chemical production with atmospheric transport. In this work, special attention is given to the hyperbolic component of the governing reaction–diffusion–advection equation (RDA), namely the advection term, which dominates the propagation of pollutant plumes and strongly influences numerical stability and accuracy.

Chemical interactions are modeled by a system of ODEs, while the evolution of PM2.5 concentration is governed by an RDA equation. The ODE system is solved by schemes suited for stiff systems, and the PDE is discretized with a Finite Volume scheme of the family of ADER method (Toro, 1999), designed to treat the hyperbolic flux with minimal numerical diffusion and spurious oscillations.

The presentation highlights several hyperbolic features: the formation of sharp concentration fronts, sensitivity to the CFL condition, and the necessity of reconstruction techniques to maintain stability while capturing steep gradients. Numerical experiments in one spatial dimension illustrate how advection regulates the structure of PM2.5 dynamics, providing insight into realistic transport phenomenon in Temuco.

Keywords: PM2.5, hyperbolic transport, advection, reaction–diffusion–advection equations, finite volume methods, atmospheric modeling.

*E-mail: `j.gonzalez44@ufromail.cl`
†E-mail: `gino.montecinos@ufrontera.cl`

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Two-layer gas-liquid flows in pipes with general cross sections

Gerardo Hernández-Dueñas *
Instituto de Matemáticas Unidad Juriquilla, Universidad Nacional Autónoma de México

Abstract

We present a new model for two-layer flows in pipelines. The lower layer consists of an incompressible liquid governed by the shallow water equations, while the upper layer is a compressible gas that exchanges momentum with the liquid phase. Our work extends the model developed in [1] by considering ducts with general cross sections and by assuming an incompressible bottom-layer fluid.

During the talk, we will present the derivation of the model and describe its hyperbolic properties. We also propose a semi-discrete, non-oscillatory, high-resolution central-upwind scheme for the numerical approximation of solutions. Time integration is performed using a second-order, strong-stability-preserving Runge–Kutta method. In addition to providing a detailed description of the scheme and demonstrating that it is well-balanced and positivity-preserving, we present several numerical experiments illustrating the robustness of the algorithm.

This is joint work with Sarswati Shah.

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*This author was supported by grant UNAM-DGAPA-PAPIIT IN115925. E-mail: `hernandez@im.unam.mx`

A path-conservative scheme based on ENO-ET reconstruction for the Saint-Venant-Exner model

Jaime Hernández Bascur *
Departamento de Ingeniería Matemática
Universidad de la Frontera

Abstract

This work presents a high-order path-conservative finite volume scheme for the numerical approximation of one-dimensional bedload sediment transport in shallow water flows, described by the Saint-Venant–Exner system. Due to the presence of non-conservative terms and the difficulties in obtaining the full eigenstructure of the system for the wide variety of bedload transport models, a centered path-conservative scheme is appropriate. The proposed method consists of a high-order extension of the centered PRICE-C path-conservative scheme [2], through the ADER approach and the novel nonlinear ENO-ET reconstruction [1]. For stability purposes, we propose approximations for the eigenvalues of the system, based on a perturbation approach specifically for the Grass bedload transport model. The resulting scheme, referred to as ADER-PRICE, is validated up to fourth-order accuracy through a set of numerical tests, including convergence rate evaluations, preservation of C-property or lake at rest condition, well-balancing analysis, long-time sediment transport simulations, and classical Riemann problems such as the dam-break problem and near-critical flow test. In parallel, a comparison between the numerical scheme combined with ENO-ET and classical ENO reconstructions up to fourth-order is performed. The results confirm the robustness, accuracy, and suitability of the proposed scheme for morphodynamic modeling over erodible beds, with the second-order scheme using ENO-ET performing particularly well in long-time simulations.

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*This author was supported by ANID Beca de Magister Nacional 2024 - 22241521). E-mail: `jaime.hernandez@ufrontera.cl`

Control for gravity waves

Rodrigo Lecaros Lira *
Departamento de Matemática, Universidad Técnica Federico Santa María.

Abstract

This presentation offers a comparative analysis of the approximate controllability properties for a range of nonlocal hyperbolic operators arising from the water-waves theory. These systems stems from the spatial non-locality inherent to the free surface dynamics. First, we present the approximate controllability of the two-dimensional sloshing problem, which models fluid oscillations within a confined (bounded) domain. The methodology centers on establishing unique continuation results for the associated non-local system. This property is shown to be sufficient to guarantee controllability via a source control localized in an interior domain. Second, we consider the control of gravity waves in the deep-water regime, modeled by a non-local hyperbolic Burgers-Hilbert equation type ($\partial_{tt}\phi + gH\partial_x\phi = 0$) on an unbounded spatial domain ($\mathbb{R}$). Our analysis verifies that this system is approximately controllable but not exactly controllable. Crucially, the unbounded nature of the domain necessitates a different control approach, utilizing a source mobile control to effectively drive the solution to zero.

This presentation is based on joint research with

- M. Fontelos, ICMAT-CSIC, U. Autónoma de Madrid, Spain.
- J. López-Ríos, U. Industrial de Santander, Colombia.
- A. Pérez, U. Bío-Bío, Chile.
- S. Zamorano, U. de Deusto, Spain.

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*This author was supported by ANID (Chile) through Centro de Modelamiento Matemático (CMM; BASAL project FB210005). E-mail: `rodrigo.lecaros@usm.cl`

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Variants for the WENO-DK of even orders for hyperbolic PDE's

Gino Montecinos Departamento de Ingeniería Matemática, Universidad de la Frontera

Abstract

The WENO-DK reconstruction, [1], is a type of WENO procedure in which, for the one-dimensional case only the leftmost, centered and rightmost stencils are involved, instead of classical WENO where all stencils around a cell of interest are required. For even orders the central WENO-DK stencil contains more elements than degrees of freedom and an overdetermined system is solved by means of a least-squares approach.

The amount of elements of stencil have an impact of the central stencil, we investigate this for even orders. Furthermore, we propose alternatives in which the solution of overdetermined systems is not required. Instead, the use of candidate central stencils within the WENO-DK approach, improves the performance of the reconstruction procedure, when applied for hyperbolic PDE's. We assess the methods in the framework of ADER method in Finite Volume and Discontinuous Galerkin Finite element approach.

Implementations of the proposed approaches in the framework of fully discrete ADER schemes for the linear advection equation and the Euler equations of gas dynamics are reported, [2]. Comparisons with conventional WENO and conventional WENO-DK confirm that the proposed variants of WENO-DK are a suitable compromise between simplicity and accuracy in the context of ADER schemes.

This presentation is based on joint research with Eleuterio Toro, University of Trento, Italy).

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A multilayer shallow water model for polydisperse reactive sedimentation

Víctor Osores *

Departamento de Matemática, Física y Estadística, Universidad Católica del Maule

Abstract

A three-dimensional model of polydisperse reactive sedimentation is developed by means of a multilayer shallow water approach. The model consists of a variety of solid particles of different sizes and densities, and substrates diluted in water, which produce biochemical reactions while the sedimentation process occurs. Based on the Masliyah-Lockett-Bassoon settling velocity, compressibility of the sediment and viscosity of the mixture, the system of governing equations is composed by non-homogeneous transport equations, coupled to a momentum equation describing the mass-average velocity. Besides, the free-surface depicted by the total height of the fluid column is incorporated and fully determined through the multilayer approach. A finite volume numerical scheme on Cartesian grids is proposed to approximate the model equations. Numerical simulations of the denitrification process exemplify the performance of the numerical scheme and model under different scenarios and bottom topographies.

This presentation is based on joint research with Julio Careaga (University of Groningen).

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*This author was supported by UCM-IN-24200 internal grant. E-mail: `vosores@ucm.cl`

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Stability and Turing Pattern Formation in a Native Prey–Predator System with an Exotic Predator: Simple Diffusion and PDE Numerical Methods

Oscar Rubilar Parra
CI^{2}MA and Departamento de Ingeniería Matemática,
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción, Chile

Abstract

In this work, we study the formation of Turing patterns in a reaction–diffusion system with simple diffusion, based on the ecological model introduced in [6] to describe the interaction among a native prey, a native predator, and an exotic predator. Starting from the interior coexistence equilibrium, we linearize the system and derive the characteristic equation as a function of the wave number κ. This allows us to analyze the stability of the equilibrium and to identify the conditions under which diffusion alone can generate spatial instabilities.

The theoretical analysis is supported by numerical simulations carried out with explicit methods and Patankar-type schemes. A systematic exploration of the parameter space reveals spatial patterns consistent with the Turing instability theory for three-species systems described in [5].

To illustrate the modeling approach, we also include simulations considering the American mink (*Mustela vison*). This invasive species poses a threat to native fauna and has been linked to declines in several prey groups, including birds and mammals, as noted in [1, 2]. It has also produced negative impacts on native competitors, such as *Lutra lutra* in Europe and *Lontra provocax* in South America, according to [3, 4].

This work is part of the requirements for the presenter's doctoral thesis seminar, which is carried out under the supervision of Dr. Raimund Bürger (Universidad de Concepción) and Dr. Luis Miguel Villada (Universidad del Bío-Bío).

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Well-posedness and numerical analysis of an elapsed time model with strongly coupled neural networks

Mauricio Sepúlveda *
CI^{2}MA & Departamento de Ingeniería Matemática
Universidad de Concepción

Abstract

The elapsed time equation is an age-structured model that describes the dynamics of interconnected spiking neurons through the elapsed time since the last discharge, leading to many interesting questions on the evolution of the system from a mathematical and biological point of view. In this work, we deal with the case when the transmission after a spike is instantaneous and the case with a distributed delay that depends on the previous history of the system, which is a more realistic assumption. Since the instantaneous transmission case is known to be ill-posed due to non-uniqueness or jump discontinuities, we establish a criterion for well-posedness to determine when the solution remains continuous in time, through an invertibility condition that improves the existence theory under more relaxed hypothesis on the nonlinearity, including the strongly excitatory case. Inspired in the existence theory, we adapt the classical explicit upwind scheme through a robust fixed-point approach and we prove that the approximation given by this scheme converges to the solution of the nonlinear problem through BV-estimates and we extend the idea to the case with distributed delay. We also show some numerical simulations to compare the behavior of the system in the case of instantaneous transmission with the case of distributed delay under different parameters, leading to solutions with different asymptotic profiles.

This presentation is based on joint research with Nicolás Torres (Université Côte d'Azur., France) and Luis Miguel Villada (Universidad del Bío-Bío, Concepción).

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*This author is supported by ANID (Chile) through projects Fondecyt 1220869, and Centro de Modelamiento Matemático (CMM; BASAL project FB210005). E-mail: mauricio@ing-mat.udec.cl

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Interaction of jamitons in second-order macroscopic traffic models

Sebastián Tapia-Mandiola *
Centre Inria de l'université de Lille

Abstract

Jamitons are self-sustaining traveling waves that arise in second-order macroscopic traffic models first introduced in [1]. In this talk, we numerically analyze the pairwise interactions of Jamitons on a circular road, motivated by the collision dynamics of traveling waves in other equations (e.g., the ϕ^4 equation and the nonlinear Schrödinger equation). First, we briefly review models known to support these waves, such as the Payne-Whitham (PW) and inhomogeneous Aw-Rascle-Zhang (ARZ) systems, together with the conditions of emergence of Jamitons and the traveling-wave construction. We then present a first-order finite-volume scheme for the inhomogeneous Aw–Rascle–Zhang model, which is validated by testing against theoretical Jamitons, and replicating behaviors observed in the literature. Because Jamitons connect distinct states through an internal shock, a compatibility condition must be imposed to identify pairs that can interact. From that, we select a wide range of compatible Jamitons and perform several experiments of different collisions between larger and smaller waves. Our results show that collisions produce a single outgoing Jamiton whose propagation speed differs from that of the initial waves and tends to smooth them. Over broad ranges of sonic densities, the outgoing amplitude and maximum density are greater than or equal to those of the incoming waves, whereas the outgoing length is strictly larger. Additional experiments, varying the relaxation time (driver reaction time), indicate that the exit speed, amplitude, and maximum density are essentially independent of this parameter, while the outgoing length scales with it.

This presentation is based on joint research [2] with Claudio Muñoz (CMM - Universidad de Chile) and Raimund Bürger ($\mathrm{CI^2MA}$ -Universidad de Concepción).

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*E-mail: `sebastian.tapia-mandiola@inria.fr`

Foam front dynamics in a porous medium: comparing Darcy flow and pressure-driven growth

Carlos Torres-Ulloa *
Departamento de ciencias Matemáticas y Físicas
Universidad Católica de Temuco

Abstract

Foam flow in a homogeneous porous medium is analyzed, focusing on computing the propagating foam front location in two-dimensions. The pressure-driven growth model [1], and a two phase Darcy's model coupled with fractional flow theory for water/gas mass conservation [2] are used and compared. The surfactant alternating gas process is studied, where foam is generated in the region where the gas meets the liquid, which then displaces the liquid phase within the porous media [3]. In this region foam exhibits a finely-textured structure, causing the foamed gas mobility to be orders of magnitude lower than that of a pure gas or pure liquid [4]. Although pressure-driven growth usually assumes this region to be a fraction ϵ of the distance travelled by the foam [1, 3, 5], a recent study suggests that ϵ is actually close to unity [6]. In order to determine whether the foam front location can be predicted accurately using the pressure-driven growth model, despite ϵ being larger than expected, predictions are compared with the aforementioned Darcy model. In addition to the foam resistance at the front, to match Darcy's predictions, it is necessary to account for the liquid resistance downstream of the front, and the fact that Darcy's predictions access liquid saturations with lower mobilities than the pressure-driven growth model considers by construction via following fractional flow theory. To achieve agreement between the models, the foamed gas mobility in pressure-driven growth model is treated as a fitting parameter, which indirectly modifies the relative resistance of the liquid phase downstream. Consequently, adjustments to the foamed gas mobility are necessary for different horizontal extents of the solution domain to ensure model compatibility and accurate foam front prediction.

This presentation is based on joint research with P. Grassia (TU Eindhoven), J. Hernández-Montelongo (UCTemuco), Y. Boakye-Ansah (UENR) and P. Zuñiga (UCTemuco).

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*This author is supported by ANID (Chile) through projects Fondecyt 3240010. E-mail: `carlos.torres@uct.cl`

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A numerical scheme for a model of a flotation column including the transport of liquid components

Yolanda Vásquez*
Universidad Tecnológica de Panamá
Centro de Estudios Multidisciplinarios en Ciencia, Ingeniería y Tecnología (CEMCIT-AIP)

Abstract

Froth flotation in a column is a key unit operation in mineral processing and wastewater treatment. A recently proposed three-phase model for a flotation column [2] describes the evolution of the volume fractions of bubbles and hydrophilic solid particles by a pair of degenerate parabolic equations with discontinuous flux, and captures the formation of a stable froth layer, the effect of wash water and the influence of variable cross-sectional area. Earlier related models and numerical schemes for three-phase flows and flotation with sedimentation can be found in [1, 3, 4].

In this work we extend that framework by incorporating the transport of several components within the liquid phase (such as slimes or different water fractions). This is achieved by coupling the original PDE system with a linear balance law for a vector of percentages attached to the liquid phase. On the numerical side, we adapt and extend the monotone conservative finite difference scheme developed in [3] so that it also approximates the evolution of the liquid components. Under a suitable Courant–Friedrichs–Lewy (CFL) condition, the scheme preserves key structural properties of the continuous model: nonnegativity of all volume fractions, an invariant-region property ensuring that the sum of the three phase volume fractions does not exceed one, and that the liquid-component percentages remain between zero and one and add up to one.

We finally use this numerical framework to construct and explore steady states of the flotation column, including regimes with "positive bias" (net downward wash-water flow through the froth). Numerical simulations illustrate how changes in operating conditions affect the existence of desirable steady states and the internal distribution of liquid components.

This presentation is based on joint work with Raimund Bürger (Universidad de Concepción), Stefan Diehl (Lund University) and María del Carmen Martí (Universitat de València).

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A nonlocal upstream-downstream traffic model

Luis-Miguel Villada *

GIMNAP-Departamento de Matemática, Universidad del Bío-Bío,
and CI^{2}MA -Universidad de Concepción

Abstract

In this talk we analyze the well-posedness of a new class of macroscopic vehicular traffic model propoded in [4] which is described by a scalar nonlocal conservation law that simultaneously incorporates both upstream and downstream effects in the flow dynamics. Unlike nonlocal models previously described in the literature, which only account for downstream density averages (look-ahead behavior), the proposed model introduces an additional term depending on an upstream average (look-behind), allowing for a more realistic representation of anticipatory driver behavior under high-density conditions. Inspired by the multiplicative flux proposed in [I. Karafyllis, D. Theodosis, and M. Papageorgiou. Analysis and control of a non-local pde traffic flow model. International Journal of Control, 95(3):660–678, 2022], our model generalizes and adapts such ideas to an entropy weak solution framework, allowing for the presence of discontinuities and shock waves. The considered flux takes the form $\rho\, g(\rho)\, W(\hat{R}_\delta)\, V(R_\eta)$, where the nonlocal terms $\hat{R}_\delta$ and R_η represent backward- and forward-looking spatial averages of the density, respectively, and the functions W and V encode the drivers' responses to these observations. The main novelty of this work lies in establishing the existence and uniqueness theory for entropy weak solutions, together with a rigorous proof of Lipschitz continuous dependence of solutions not only on the initial data, but also on the kernel functions, under reasonable structural assumptions on the flux components. The proofs are achieved through the design of a conservative numerical scheme that preserves key structural properties of the continuous model, such as maximum principle, mass conservation, $\mathbf{BV}$ estimates, and $\mathbf{L}^1$-stability. Finally, we present numerical experiments that illustrate the behavior of solutions and the qualitative impact of nonlocal terms on traffic dynamics.

This presentation is based on joint research with Harold Contreras (USS-Chile) and Paola Goatin (INRIA-Sophia-Antipolis, France).

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